Checking The Averaging Eqn for VMTL G

Introduction

Suppose G is a **tree** that admits a **vertex magic total labeling (VMTL)** with magic constant h.
Let:

- ℓ: number of **leaf vertices**
- n: number of **non-leaf (internal) vertices**
- Total number of vertices: $n + \ell$
- Number of edges in a tree: $n + \ell - 1$
- Labels are assigned from 1 to maximum value $M = 2n + 2\ell - 1$
- S_v: sum of all **vertex labels**
- S_e: sum of all **edge labels**
- $S_{v\ell}$: sum of **labels on leaf vertices**
- $S_{e\ell}$: sum of **labels on edges incident to leaves**
- S_{vo}: sum of **non-leaf vertex labels**, $S_{vo} = S_v - S_{v\ell}$
- S_{eo}: sum of **non-leaf edge labels**, $S_{eo} = S_e - S_{e\ell}$
- S_{eo4}: sum of **central edges incident with degree-4 vertices**
- S_{eod}: sum of **edges incident to degree-3 vertices with one leaf**, $S_{eod} = S_{eo} - S_{eo4}$
- $d = 2n - \ell$: number of **degree-3 vertices with one leaf**

Total Label Sum

The total sum of all labels is the sum of the first $M = 2n + 2\ell - 1$ natural numbers:

$$S_v + S_e = \sum_{i=1}^{M} i = \frac{M(M+1)}{2}$$

Substitute $M = 2n + 2\ell - 1$:

$$S_v + S_e = (n + \ell)(2n + 2\ell - 1) \tag{1}$$

Each vertex weight is the sum of its own label and the labels of its incident edges. Since each edge is counted twice (once at each end), we get:

$$S_v + 2S_e = (n + \ell)h \tag{2}$$

Deriving Vertex and Edge Label Sums

Subtract Equation (2) from Equation (1):

$$(S_v + S_e) - (S_v + 2S_e) = (n + \ell)(2n + 2\ell - 1 - h) \Rightarrow -S_e = (n + \ell)(2n + 2\ell - 1 - h)$$

Solving:

$$S_e = (n + \ell)(h - (2n + 2\ell - 1))$$

From Jonathan's theorem:

$$h = 4n + 2\ell - 1 \Rightarrow S_e = (n + \ell)(2n) \Rightarrow \boxed{S_e = 2n(n + \ell)}$$

Then:

$$S_v = (n + \ell)(2n + 2\ell - 1) - 2n(n + \ell) = (n + \ell)(2\ell - 1) \Rightarrow \boxed{S_v = (n + \ell)(2\ell - 1)}$$

Leaf Vertex and Edge Sums

We assume that the largest labels are assigned to the leaf vertices. Since vertex labels range from 1 to $M = 2n + 2\ell - 1$, the leaf vertex labels will range from M down to $M - \ell + 1$. This is an arithmetic series, so the sum is:

$$S_{v\ell} = \frac{\ell}{2}[2M - (\ell - 1)] = \frac{\ell}{2}(4n + 3\ell - 1)$$

From Jonathan's paper, we know:

$$h = 4n + 2\ell - 1$$

We try to label the edges incident to leaves such that they follow the values:

$$h - M,\ h + M - 1,\ h + M - 2,\ \ldots,\ M - \ell$$

This becomes an arithmetic progression starting from $2n$ to $2n + \ell - 1$, hence:

$$S_{e\ell} = 2n + (2n + 1) + \cdots + (2n + \ell - 1) = \frac{\ell}{2}(4n + \ell - 1)$$

Now we find the remaining (non-leaf) sums:

$$S_{vo} = S_v - S_{v\ell} = (n + \ell)(2\ell - 1) - \frac{\ell}{2}(4n + 3\ell - 1) = \frac{\ell^2}{2} - \frac{\ell}{2} - n \Rightarrow \boxed{S_{vo} = \frac{\ell^2}{2} - \frac{\ell}{2} - n}$$

$$S_{eo} = S_e - S_{e\ell} = 2n(n + \ell) - \frac{\ell}{2}(4n + \ell - 1) = 2n^2 - \frac{\ell^2}{2} + \frac{\ell}{2} \Rightarrow \boxed{S_{eo} = 2n^2 - \frac{\ell^2}{2} + \frac{\ell}{2}}$$

Edge Allocation Strategy

Assume that the central edges incident with degree-4 vertices receive the smallest edge labels, leaving the highest values for more constrained vertices (e.g., degree-3 vertices with one leaf).

We have a total of $n - 1$ central edges.

The number of edges incident with degree-4 vertices is:

$$n - 1 - d$$

Let S_{eo4} be the sum of these smallest edge labels. Since these labels form the initial segment of natural numbers, the sum is:

$$S_{eo4} \geq 1 + 2 + \cdots + (n - 1 - d) = \frac{(n - 1 - d)(n - d)}{2}$$

Then the remaining edge sum used by the d constrained vertices (degree-3 with one leaf) is:

$$S_{eod} = S_{eo} - S_{eo4}$$

Numerical Example: $n = 41$, $\ell = 71$

Evaluate:

$$M = 2n + 2\ell - 1 = 223, \quad h = 4n + 2\ell - 1 = 305$$

$$S_v = 112 \cdot 141 = 15{,}792, \quad S_e = 112 \cdot 82 = 9{,}184$$

$$S_{v\ell} = \frac{71}{2}(223 \cdot 2 - 70) = \frac{71}{2}(446 - 70) = \frac{71}{2}(376) = 13{,}348$$

$$S_{e\ell} = \frac{71}{2}(164 + 70) = \frac{71}{2}(234) = 8{,}307$$

$$S_{vo} = 15{,}792 - 13{,}348 = 2{,}444, \quad S_{eo} = 9{,}184 - 8{,}307 = 877$$

$$d = 82 - 71 = 11, \quad \text{Degree-4 edge count} = 41 - 1 - 11 = 29$$

$$S_{eo4} = \frac{29 \cdot 30}{2} = 435 \Rightarrow S_{eod} = 877 - 435 = 442$$

We attempt to assign the 40 largest available distinct edge labels from 1 to 81, summing to 877. A program outputs:

$$\{81, 55, 38, 37, 36, 35, 34, 33, 32, 31, 30, 29, \ldots, 1\}$$

Assigning the smallest values to the degree-4 vertices, the remaining 11 edge labels used by the constrained degree-3 vertices with one leaf are:

$$\{81, 55, 38, 37, 36, 35, 34, 33, 32, 31, 30\}$$

Edge Assignment and Averaging Equation

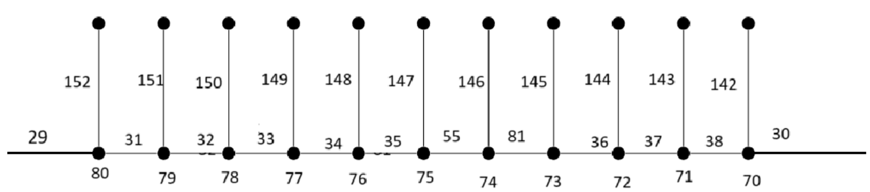

Figure 1: Edge Labeling Assignment for Averaging Equation

Each edge is counted twice in the vertex weight, except endpoints. The total vertex weight for the 11 degree-3 vertices with one leaf is:

$$\text{Total Weight} = 3{,}325 \Rightarrow \text{Average Weight} = \frac{3{,}325}{11} = 302.27$$

Since this average is below the required $h = 305$, even under optimal allocation, we conclude:
No VMTL exists for $n = 41$, $\ell = 71$.